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Extensive scaling and nonuniformity of the Karhunen-Loève decomposition for the spiral-defect chaos state

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By analyzing large-aspect-ratio spiral-defect chaos (SDC) convection images, we show that the Karhunen-Loève decomposition (KLD) scales extensively for subsystem sizes larger than $4d$ (d is the fluid depth), which strongly suggests that SDC is extensively chaotic. From this extensive scaling, the intensive length ξ_{KLD} is computed and found to have a different dependence on the Rayleigh number than the two-point correlation length ξ_2. Local computations of ξ_{KLD} reveal a spatial nonuniformity of SDC images that extends over radii $18d<r<45d$ in a $\Gamma=109$ aspect-ratio cell. [S1063-651X(98)50412-6]

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A significant theoretical challenge is to find ways to characterize the nonperiodic time-dependent patterns often observed in large sustained nonequilibrium systems [1]. In a recent paper [2], a length scale for characterizing spatiotemporal chaos (STC), the KLD length ξ_{KLD}, was proposed based on the *extensive scaling* of the Karhunen-Loève decomposition (KLD) [3]. This length scale was shown [2] to be computable from moderate amounts of space-time data and was shown to contain information similar to the fractal-dimension density, which has not yet been computed from experimental data [4]. For certain idealized mathematical models, the length ξ_{KLD} was shown to have a different parametric dependence than the commonly computed two-point correlation length ξ_2 and therefore provides a different way to characterize spatiotemporal chaos [2]. Further, this length ξ_{KLD} could be calculated from data localized to a region of space and therefore a way to analyze spatial inhomogeneities. However, to the best of our knowledge, an application of the KLD length to *experimental* data had not been made prior to this work.

In this paper we present the first application of the KLD length to experimental STC data by analyzing the recently discovered [5] spiral-defect chaos (SDC) in Rayleigh-Bénard convection. We analyze the SDC state of large-aspect-ratio ($\Gamma\equiv r/d=29$ and 109, where r and d are the radius and thickness of the cell) cells and find that the KLD dimension D_{KLD} of the data scales extensively with subsystem volume when the diameter of the subvolume is larger than about $4d$, where d is the depth of the fluid. This extensivity strongly suggests that the SDC state is extensively chaotic [4,6], which provides the first such evidence of an experimental system. From the extensive scaling of the dimension D_{KLD} with subvolume, we calculate the length ξ_{KLD}, which reflects the density of linearly independent modes needed to approximate the spatiotemporal data [2]. For the reduced Rayleigh numbers ϵ explored in this study, ξ_{KLD} exhibits a minimum with increasing ϵ while the two-point correlation length ξ_2 monotonically decreases. We speculate that this different behavior in the KLD length arises from changes in the structure of spirals and straight rolls with increasing ϵ. Finally, we also show that the length ξ_{KLD} provides a way to measure an inhomogeneity in the SDC data within the interior of the cell. A local KLD analysis demonstrates that the data are not uniform. (Most numerical simulations of SDC [7] have used periodic lateral boundary conditions, for which the dynamics should be statistically homogeneous by translational invariance.) In addition, this nonuniformity extends to within a radius $r=18d$ of the center of an aspect-ratio $\Gamma=109$ cell. These results suggest that the length ξ_{KLD} can provide useful insight into spatiotemporal chaos.

We tested the utility of the length ξ_{KLD} for analyzing SDC experimental data [5,8] by collecting thousands of shadowgraph images [9] of the SDC state from a Rayleigh-Bénard experiment using compressed CO_2 gases with Prandtl number $\sigma\approx 1$, and in cylindrical cells of aspect ratio $\Gamma=29$ and 109. SDC appeared for $\epsilon\equiv\Delta T/\Delta T_c-1$ above 0.56 and 0.23, respectively, for the $\Gamma=29$ and 109 cells, where ΔT_c is the critical temperature difference for the onset of convection. Five sets of SDC data were taken in the two cells as summarized in Table I. We chose sampling times Δt that were significantly longer than the correlation time $\tau\sim 10t_v$ (where t_v is the vertical thermal diffusion time) to maximize the amount of uncorrelated data [5]. Good spatial resolution of the convection patterns required at least five pixels per convection roll for the $\Gamma=109$ cell. Data were taken from the central 43% of the $\Gamma=109$ convection cell, as indicated in Fig. 1.

To illustrate the KLD analysis used to study the experimental data, let $u(t_i,\mathbf{x}_j)$ denote the light intensity of an experimental image at position $\mathbf{x}_j$ at time t_i. The analysis proceeds by constructing a $T\times S$ space-time matrix of data,

$$A_{ij}=u(t_i,\mathbf{x}_j)-\langle u(t_i,\mathbf{x}_j)\rangle, \tag{1}$$

where $\langle u(t_i,\mathbf{x}_j)\rangle$ denotes the time average (average over index i) of the set of experimental images $u(t_i,\mathbf{x}_j)$, T is the

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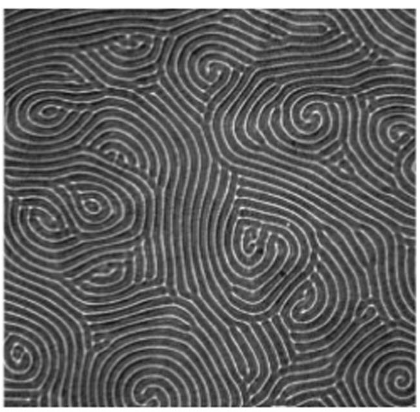
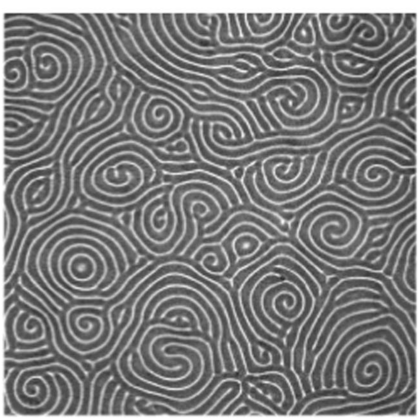
TABLE I. Parameters for data used to calculate $\xi_{\rm KLD}$. Γ is the aspect ratio, ϵ is the reduced Rayleigh number, T is the number of images, and $\Delta t/t_v$ is the sampling rate in units of the vertical thermal diffusion time t_v.

Γ	ϵ	T	$\Delta t/t_v$	t_v (s)
29	1.80	2048	33	8.28
109	0.52	3100	1050	0.86
109	0.67	2500	1050	0.86
109	0.79	2500	1050	0.86
109	0.93	2570	728	0.82

number of observation times t_i, and S is the number of observation sites $\mathbf{x}_j$. The KLD dimension $D_{\rm KLD}$ [10] of the matrix A_{ij} then measures the number of linear eigenmodes needed to approximate some fraction $0<f<1$ of the variance of the experimental data and can be computed from the eigenvalues of the matrix $\mathbf{A}^T\mathbf{A}$ [2,10,11]. We compute $D_{\rm KLD}(\mathbf{x}_j)$ for concentric subsystems of volume V (square or circular geometry) that are centered at a particular point $\mathbf{x}_j$ in space. $D_{\rm KLD}(\mathbf{x}_j)$ depends on the point $\mathbf{x}_j$ and therefore provides a measure of dynamical inhomogeneity. If $D_{\rm KLD}(\mathbf{x}_j)$ increases linearly with subsystem volume V and with a slope δ, then the length $\xi_{\rm KLD}$ is defined to be $\delta^{-1/d}$, where d is the dimensionality of the data ($d=2$ for SDC).



(a)



(b)

FIG. 1. Time-dependent shadowgraph patterns for the central 43% of the $\Gamma=109$ convection cell, with light and dark regions corresponding to cool and warm fluids, respectively. (a) and (b) are snapshots of the SDC pattern for reduced Rayleigh numbers $\epsilon=0.52$ and $\epsilon=0.93$, respectively.

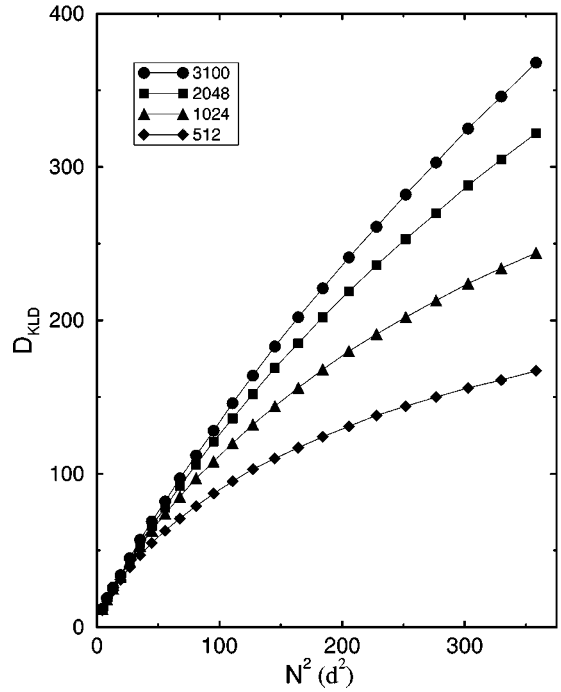


FIG. 2. Scaling of the KLD dimension $D_{\rm KLD}$ with subsystem area N^2 for data in the central 43% of the $\Gamma=109$ cell with reduced Rayleigh number $\epsilon=0.52$ and variance fraction $f=0.70$. The area N^2 of a subsystem is measured in units of d where d is the thickness of the cell. The labels indicate the number of images used in the calculations, and lines connecting points were drawn to guide the eye.

In applying these ideas to SDC data, we computed $D_{\rm KLD}$ for a fixed fraction $f=0.7$ [12] and for larger and larger square subimages in the center of the convection cell of size $S\times S$, where $2d<S<25d$. As shown in Fig. 2, the dimension $D_{\rm KLD}$ scales approximately linearly with subsystem data over a relatively large range of subsystem sizes $4d<S<13d$ provided that a sufficiently long time series was used. This extensive scaling, together with the arguments in Ref. [2], relating $\xi_{\rm KLD}$ to the dimension correlation length ξ_δ, strongly suggests that the SDC state is extensively chaotic. We note that the extensive linear scaling of the dimension $D_{\rm KLD}$ with the subsystem area is best for smaller subsystems, which we believe is a consequence of the fact that smaller subsystems have a faster time scale to become statistically stationary.

Next we compared $\xi_{\rm KLD}$ (computed in the center of the cell for $f=0.7$) with the two-point correlation length ξ_2 as the reduced Rayleigh number ϵ was varied. The two-point correlation length ξ_2 was calculated from the inverse of the width of the peak in the Fourier spectrum of the spatial data, which was premultiplied with a hanning window of diameter equal to the lateral dimension of the images. As shown in Table II, the parametric dependences of the length $\xi_{\rm KLD}$ and the two-point correlation length ξ_2 are different [12]. The fact that $\xi_{\rm KLD}$ attains a minimum between $\epsilon=0.79$ and 0.88 and then increases with increasing ϵ (corresponding to a decrease in complexity) is somewhat counterintuitive, since one might have expected $\xi_{\rm KLD}$ to decrease with ϵ as the

TABLE II. Lengths ξ_{KLD} and ξ_2 (normalized to the depth of the fluid d) as functions of aspect ratio Γ and reduced Rayleigh number ϵ. The length ξ_{KLD} was calculated for fraction $f=0.7$.

Γ	ϵ	ξ_{KLD}/d	ξ_2/d
29	1.80	1.47	2.28
109	0.52	0.84	4.38
109	0.67	0.81	3.85
109	0.79	0.75	3.60
109	0.88	0.74	3.39
109	0.93	0.90	3.22

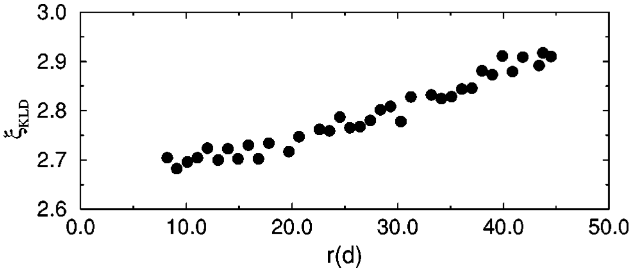


FIG. 3. Radial inhomogeneities of the length ξ_{KLD} in annular subsystems of fixed radii and increasing azimuthal angle for the $\Gamma=109$ convection cell ($\epsilon=0.79$). The radial distance r is measured in units of d (the thickness of the convection cell) from the center of the $\Gamma=109$ convection cell.

system is forced further away from equilibrium (more modes are needed to approximate the space-time data). A possible explanation for these opposing trends may be found by examining the different spatial structures in Figs. 1(a) and 1(b). For Fig. 1(a) (ϵ=0.52), the area fraction of local straight-roll regions is larger than that for spiral-roll regions, whereas for Fig. 1(b) (ϵ=0.93) the relationship between straight- and spiral-roll regions is reversed. We speculate that the data between ϵ=0.79 and 0.88 consist of nearly equal fractions of straight- and spiral-roll regions and thus require more KLD eigenmodes per unit volume $D_{\mathrm{KLD}}/V_{\mathrm{sub}}$ and so $\xi_{\mathrm{KLD}}=(D_{\mathrm{KLD}}/V_{\mathrm{sub}})^{-1/2}$ is smaller. This speculation about the relative complexity of straight-roll versus spiral-roll regions is supported by examining the KLD spatial eigenmodes for the $\epsilon=0.52$ and $\epsilon=0.93$ data. These modes have the same qualitative symmetries up to eigenmode 21 (f=0.45 for $\epsilon=0.52$), beyond which the local straight-roll regions in the $\epsilon=0.52$ data entered into the next KLD spatial modes and broadened the KLD eigenvalue spectrum [2]. In the $\epsilon=0.93$ data, beyond eigenmode 21 the eigenmodes consisted of lattices of convection rolls.

Since the KLD analysis of a subsystem is a local procedure, one can quantify differences in the SDC in different regions of the convection cell and therefore test the assumption that SDC is dynamically homogeneous. For the $\Gamma=109$ cell we investigated a radial dependence of ξ_{KLD} by first establishing that D_{KLD} scaled extensively for an annulus of radius $r_o<r<r_o+d$ and an angular sector that varied from $\Delta\theta=0$ to $\Delta\theta=2\pi$ radians (extensivity was with respect to $\Delta\theta$). We could not estimate nonuniformity for $r_o<8d$ because the subsystems were too small, or for $r_o>45d$ as this exceeded the area imaged by the charge-coupled-device camera. Figure 3 shows how the length ξ_{KLD} increases by 10% from $r=8d$ to $r=45d$. The variation in the length ξ_{KLD} with radial distance r demonstrates that the images of SDC cannot be considered homogeneous (Fig. 3) even in large-aspect-ratio cells. Based on this calculation we consider the cell *approximately* uniform for $r<18d$. The 10% nonuniformity of ξ_{KLD} in the large cell may be due to dynamical and/or experimental reasons. The dynamical reason would be due to the pinning of the pattern by the side wall, which has been shown to cause time-averaged patterns near the side wall, for instance, in rotating convection experiments [13]. (We also found nonuniformity of ξ_{KLD} near the side wall in the $\Gamma=29$ cell.) The experimental source could come from the nonlinearity of the shadowgraph method. For thin cells and high ϵ values, the nonlinearity is strong and can make the shadowgraph method sensitive to small optical nonuniformities. Unfortunately, the relative strength of these two sources of nonuniformity cannot be determined.

In summary, we have carried out an analysis of experimental data using the extensive scaling of KLD in subsystems [2] for the spiral-defect chaos state. KLD analysis is straightforward to apply to small subsystems of different geometry since it does not impose an approximate periodicity of the space-time data, as is the case for Fourier analysis. By verifying that the dimension D_{KLD} scaled linearly with subsystem size, we provide strong evidence that the experimental system is extensively chaotic [1]. The utilization of a subsystem was essential for our study for two reasons. First, local analysis of subsystems allowed the characterization of an inhomogeneity in our experimental convection cell. Further, by exploiting the reduced data requirements of subsystems, we could estimate the parametric behavior of the dimension density (the average number of degrees of freedom per unit area) in the center of the convection cell. For the SDC data analyzed, the length ξ_{KLD} seems to quantify differences in the fraction of straight-roll and spiral-defect regions (also observed in the KLD spatial eigenmodes), and therefore provides information beyond that available from ξ_2. Finally, the nonuniformity of the KLD length was shown to extend over 80% of the radius in the $\Gamma=109$ convection cell. Future analysis will hopefully provide further insight into the transition to SDC and the chiral symmetry breaking recently observed in SDC for rotating convection experiments [14].

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